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M. A. (Part - II) External - Examination April - 2026

Mathematics (Paper - 5002)

Advanced Integral Transforms

Time: 3 Hours] **[Total Marks: 100**

સૂચના : / Instructions	Seat No.:
(1) ઉપરોક્ત દર્શાવેલ વિગતો ઉત્તરવહી પર અવશ્ય લખવી. Fill up strictly the above details on your answer book	
(2) Answer all questions.	
(3) Figure to the right indicates marks.	
(4) Follow usual notations and conventions.	Student's Signature

Q.1 (a) In usual notation prove that **07**

$$\bar{f}_n''(s) = \frac{s^2}{4}\left[\left(\frac{n+1}{n-1}\right)\bar{f}_{n-2}(s) - 2\left(\frac{n^2-3}{n^2-1}\right)\bar{f}_n(s) + \left(\frac{n-1}{n+1}\right)\bar{f}_{n+2}(s)\right]$$

(b) The vibration of very large membrane is generated by the equation **07**

$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r}\frac{\partial U}{\partial r} = \frac{1}{c^2}\frac{\partial^2 U}{\partial t^2}$, r ≥ 0 and t ≥ 0. Prove that for $t > 0$,

$$U(r,t) = \int_0^\infty p\bar{f}(p)\cos(p(t))J_0(pr)dp + \frac{1}{c}\int_0^\infty \bar{g}(p)\sin(p(t))J_0(pr)dp\ ,$$

where $J_0(pr)$ is Bessel's function of order zero with conditions

$U = f(r), t = 0$ and $\frac{\partial U}{\partial t} = g(r), t = 0$.

(c) Find (i) $H^{-1}[s^{-2}e^{-as},\ n = 1]$ **06**

(ii) $H^{-1}[e^{-as},\quad n = 0]$

OR

Q.1 (a) Show that Hankel transform of order zero of the function $f(x)$ defined by **07**

$f(x) = \begin{cases} a^2-x^2, & 0 < x < a \\ 0, & x > a \end{cases}$ is given by $\frac{4a}{s^3}J_1(as) - \frac{2a^2}{s^2}J_0(as)$

(b) Find Hankel transform of $\frac{d^2f}{dx^2} + \frac{1}{x}\frac{df}{dx} - \frac{n^2}{x^2}f$ taking $xJ_n(sx)$ as a kernel. **07**

(c) Define Finite Hankel transform. Find Hankel transform of $x^{n-1}, n > 1$ by taking $xJ_{n-1}(sx)$ as a kernel. **06**

Q.2 (a) Using finite Hankel transform, prove that **07**

$$\int_0^1 \frac{J_n(sx)}{J_n(a)} xJ_n(sx)dx = \frac{s}{a^2-s^2}J_n'(s),\ \text{where}\ J_n(s) = 0.$$

(b) If $H_i[f(t)] = \overline{f_H}\,(x)$, then prove that 07
(i) $H_i[f(t+a)] = \overline{f_H}\,(x+a)$
(ii) $H_i[f(-at)] = -\overline{f_H}\,(-ax)$

(c) Find Hilbert transform of (i) $f(t) = cos(\omega t)$ (ii) $f(t) = sin(\omega t)$ 06

OR

Q.2 (a) Find finite Hankel transform of $(1 - x^2)$ if $xJ_0(s_i x)$ is a kernel of the transform. 07

(b) Prove that (i) $H_i[f'(t)] = \frac{d}{dx}\left[\overline{f_H}\,(x)\right]$ 07
(ii) $H_i[tf(t)] = x\left[\overline{f_H}\,(x)\right] + \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)dt$

(c) Find Hilbert transform of rectangular pulse given by 06
$$f(t) = \begin{cases} 1\,; for\ |t| < a \\ 0\,; for\ |t| > a \end{cases}$$

Q.3 (a) Prove that $H_n[f'(x)] = \bar{f}_n'(s) = -\frac{s}{2n}\left[(n+1)\bar{f}_{n-1}(s) - (n-1)\bar{f}_{n+1}(s)\right]$. 07

(b) Solve the integral equation $\lambda \int_0^{\infty} \frac{f(t)}{t+x}\,dt = f(x)$, where λ is a real parameter by Steltjes transform. 07

(c) Find S-transform of (i) $f(t) = (t-a)^{-1}$
(ii) $f(t) = t^{\alpha-1}$ 06

OR

Q.3 (a) If $S[f(t)] = \bar{f}\,(z)$, then prove that 07
(i) $S[f(t+a)] = \bar{f}\,(z-a)$
(ii) $S[tf(t)] = -z\bar{f}\,(z) + \int_0^{\infty} f(t)dt$, provided $\int_0^{\infty} f(t)dt$ exists.
(iii) $S\left[\frac{1}{t} f(\frac{a}{t})\right] = \frac{1}{z}\,\bar{f}\,\left(\frac{a}{z}\right)$

(b) Using finite Hankel transform solve the differential equation 07
$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r}\frac{\partial V}{\partial r} = \frac{1}{k}\frac{\partial V}{\partial t}$; $0 \le r < 1$, $t > 0$ subject to conditions, $V = V_0$ (constant) for $r = 1$, $t > 0$ and $V = 0$ for $t = 0$, $0 \le r < 1$.

(c) For S-transform, prove that 06
$$S[f^{(n)}(t)] = -\left[\frac{1}{z} f^{(n-1)}(0) + \frac{1}{z^2} f^{(n-2)}(0) + \cdots + \frac{1}{z^n} f(0)\right] + (-1)^n \frac{d^n}{dz^n}\left[\bar{f}(z)\right]$$

Q.4 (a) Find Mellin transform of $f(x)=e^{-nx}, n>0$. 07

(b) In usual notation prove that $M\left[x^n f^{(n)}(x)\right]=(-1)^n \frac{\Gamma(p+n)}{\Gamma(p)}\bar{f}(p)$. 07

(c) Use Mellin transform to solve the integral equation 06

$$\int_0^\infty f(\xi)g\left(\frac{x}{\xi}\right)\frac{d\xi}{\xi}=h(x)$$

OR

Q.4 (a) If $M[f(x)]=\bar{f}(p)$, then prove that (i) $M[f(ax)]=a^{-p}\bar{f}(p)$ 07

(ii) $M[\log(x)f(x)]=\frac{d}{dp}\bar{f}(p)$

(b) Apply Mellin transform to solve the boundary value problem 07

$$x^2u_{xx}+xu_x+u_{yy}=0,\ 0\le x<\infty, 0<y<\infty$$

$$u(x,0)=0 \text{ and } u(x,1)=\begin{cases} A; 0\le x\le 1 \\ 0; x>1 \end{cases}$$

(c) Define Mellin Transform. Find Mellin transform of $f(x)=\frac{1}{e^x-1}$. 06

Q.5 (a) Prove that $Z\left(n^p\right)=-z\frac{d}{dz}Z\left(n^{p-1}\right)$, where p is any positive integer. Deduce that $Z(n)=\frac{z}{(z-1)^2}$ and $Z(n^2)=\frac{z^2+z}{(z-1)^3}$. 07

(b) Using Z-transform solve $u_{n+2}-2u_{n+1}+u_n=2^n$ subject to the condition $u_0=2, u_1=1$. 07

(c) Find the Z-transform of (i) $u_n=a^n\sin n\theta$ (ii) $u_n=a^n\cos(n\theta)$. 06

OR

Q.5 (a) Using Z-transform solve the system of difference equations 07

$$x_{n+1}-7x_n-10y_n=0$$

$$y_{n+1}-x_n-4y_n=0$$

given $x_0=3, y_0=2$.

(b) If $Z^{-1}\left[\bar{u}(z)\right]=u_n$ and $Z^{-1}\left[\bar{v}(z)\right]=v_n$, then prove that 07

$$Z^{-1}\left[\bar{u}(z).\bar{v}(z)\right]=\sum_{m=0}^{n}u_m.v_{n-m}=u_n*v_n$$

(c) Solve using Z-transform $y_{n+2}-5y_{n+1}+6y_n=1$, given that $y_0=0, y_1=1$. 06
